Caringbah High School

Year 12 2020
Mathematics Extension 2
HSC Course
Assessment Task 4

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided
- In Questions 11-16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I **10 marks**

Attempt Questions 1-10

Mark your answers on the answer sheet provided. You may detach the sheet and write your name on it.

Allow about 15 minutes for this section

Section II **90 marks**

Attempt Questions 11-16

Write your answers in the numbered answer booklets provided. Ensure your name or student number is clearly visible.

Allow about 2 hours and 45 minutes for this section.

Marker's Use Only							
Section I	**Section II**						**Total**
Q 1-10	**Q11**	**Q12**	**Q13**	**Q14**	**Q15**	**Q16**	
/10	/15	/15	/15	/15	/15	/15	/100

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10

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1. Which of the following is true for all complex numbers z?

(A) $\operatorname{Im} z = \dfrac{z - \overline{z}}{2i}$

(B) $\operatorname{Im} z = \dfrac{z + \overline{z}}{2}$

(C) $\operatorname{Im} z = \dfrac{z - \overline{z}}{2}$

(D) $\operatorname{Im} z = \dfrac{z + \overline{z}}{2i}$

2. If $\int f(x) \sin x \, dx = -f(x) \cos x + \int 3x^2 \cos x \, dx$,
which of the following could be $f(x)$?

(A) $3x^2$

(B) x^3

(C) $-x^3$

(D) $-3x^2$

3. Which of the following is $\dfrac{3x + 11}{(x - 3)(x + 1)}$ expressed in partial fractions?

(A) $-\dfrac{1}{x-3} - \dfrac{4}{x+1}$

(B) $\dfrac{5}{x-3} - \dfrac{2}{x+1}$

(C) $\dfrac{5}{x-3} + \dfrac{2}{x+1}$

(D) $-\dfrac{1}{x-3} + \dfrac{4}{x+1}$

4. Given that *x* and *y* are real numbers, which of the following statements is true?

(A) $\forall y(\exists x : |x| = y)$

(B) $\forall y(\exists x : |x| < y)$

(C) $\forall y(\exists x : |x| > y)$

(D) $\forall y(\exists x : |x| = -y)$

5. What is the magnitude of the vector $\cos\theta\,\underset{\sim}{i} + \sin\theta\,\underset{\sim}{j} + \tan\theta\,\underset{\sim}{k}$

where $0 < \theta < \dfrac{\pi}{2}$?

(A) 1

(B) $\operatorname{cosec}\theta$

(C) $\cot\theta$

(D) $\sec\theta$

6. Consider the statement $x^2 = 9 \Rightarrow x = 3$. Which of the following statements is the contrapositive?

(A) $x \neq 3 \Rightarrow x^2 \neq 9$

(B) $x^2 \neq 9 \Rightarrow x \neq 3$

(C) $x = 3 \Rightarrow x^2 = 9$

(D) $x \neq 3 \Leftrightarrow x^2 \neq 9$

7. The points *A*, *B* and *C* are collinear where
$\overrightarrow{OA} = \underset{\sim}{i} + \underset{\sim}{j}$, $\overrightarrow{OB} = 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$, $\overrightarrow{OC} = 3\underset{\sim}{i} + a\underset{\sim}{j} + b\underset{\sim}{k}$.
What are the values of *a* and *b*?

(A) $a = -3,\ b = -2$

(B) $a = 3,\ b = -2$

(C) $a = -3,\ b = 2$

(D) $a = 3,\ b = 2$

8. A particle is moving in simple harmonic motion about a fixed point O on a line. At time t seconds, its displacement from O is given by $x = 2\cos \pi t$ metres. What is the time taken by the particle to travel the first 100 metres of its motion?

(A) 20 seconds

(B) 25 seconds

(C) 50 seconds

(D) 100 seconds

9. A particle is projected horizontally with speed $\sqrt{gh}$ ms^{-1} from the top of a tower of height h metres. It moves under gravity where the acceleration due to gravity is g ms^{-2}. At what angle to the horizontal will the particle hit the ground?

(A) $\tan^{-1}\frac{1}{2}$

(B) $\tan^{-1}\frac{1}{\sqrt{2}}$

(C) $\tan^{-1}\sqrt{2}$

(D) $\tan^{-1}2$

10. The equation $z^5 = 1$ has roots $1,\ \omega,\ \omega^2,\ \omega^3,\ \omega^4$ where $\omega = e^{\frac{2\pi}{5}i}$. What is the value of $(1-\omega)\left(1-\omega^2\right)\left(1-\omega^3\right)\left(1-\omega^4\right)$?

(A) –5

(B) –4

(C) 4

(D) 5

End of Section I

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available. For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet.

(a) A particle is moving in a straight line. At time t seconds it has displacement x metres from a fixed point O on the line, where x is given by $x = 1 + \cos 2t + \sin 2t$

(i) Express x in the form $x = 1 + a\cos(2t - \alpha)$ for constants $a > 0$ and $0 < \alpha < \dfrac{\pi}{2}$. **2**

(ii) Find correct to 2 decimal places the average speed of the particle during the time it takes to first reach O. **3**

(b) In an Argand diagram the point P represents $z_1 = 3 + 2i$, the point Q represents $z_2 = \dfrac{12 - 5i}{z_1}$ and O is the origin. z_3 represents the centre C of the circle passing through P, Q and O.

(i) Express z_2 in the form $a + ib$ where a and b are real. **2**

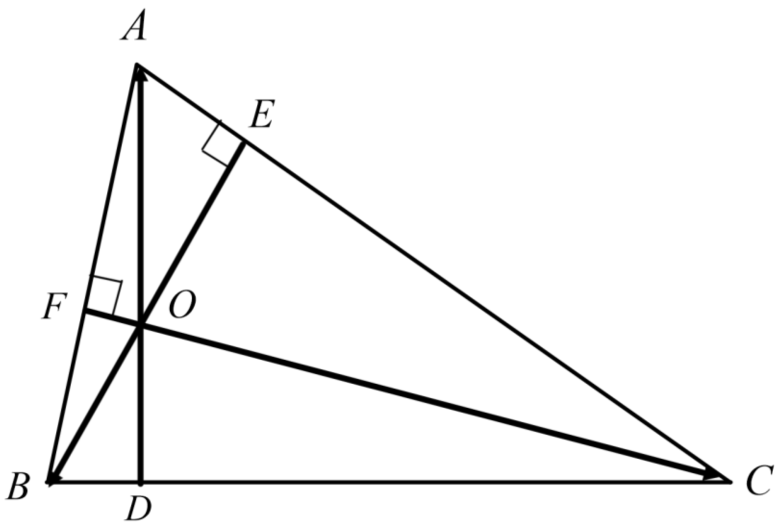
(ii) Show that $\angle POQ = \dfrac{\pi}{2}$. **1**

(iii) Express z_3 in the form $a + ib$ where a and b are real. **2**

Question 11 continues on page 6

Question 11 (continued)

(c)



ABC is an acute angled triangle. The altitudes *BE* and *CF* intersect at *O*. The line *AO* produced meets *BC* at *D*. Relative to *O* the position vectors of *A*, *B* and *C* are $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ respectively.

(i) Show that $\underset{\sim}{b}.(\underset{\sim}{c}-\underset{\sim}{a})=0$ and $\underset{\sim}{c}.(\underset{\sim}{b}-\underset{\sim}{a})=0$ **2**

(ii) Hence show that $AD \perp BC$ **2**

(iii) What geometrical property of the triangle has been proved? **1**

End of Question 11

Question 12 (15 marks) Use the Question 12 Writing Booklet.

(a) Find $\int \frac{x^2}{x^2+1}\,dx$. **2**

(b) **(i)** Use the substitution $u = x^2 - 4$ to show that **2**

$$\int \frac{x}{\sqrt{x^2-4}}\,dx = \sqrt{x^2-4} + c \ .$$

(ii) Hence find the exact value of $\int_{\sqrt{5}}^{\sqrt{8}} \frac{x\ln(x^2-4)}{\sqrt{x^2-4}}\,dx$. **3**

(c) **(i)** Find the parametric equations of the line $\underset{\sim}{l} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ passing through **2**

$A = (2, -1,\ 3)$ which is parallel to $\underset{\sim}{v} = \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix}$.

(ii) Show that the point $B = (10,\ 5, -1)$ lies on this line. **2**

(d) If $\underset{\sim}{a} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{b} = -\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$, **2**
find a unit vector perpendicular to both $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

(e) It is given that $a > 0$ and $b > 0$ are real numbers. Consider the statement **2**

$$\forall a\left(\forall b, \log_{\frac{1}{a}} \frac{1}{b} = \log_a b\right) .$$

Prove that the statement is true.

End of Question 12

Question 13 (5 marks) Use the Question 13 Writing Booklet.

(a) It is given that $z = 2e^{\frac{\pi}{12}i}$ is a root of the equation $z^4 = a\left(1+\sqrt{3}\,i\right)$, where a is real.

(i) Express $1+\sqrt{3}\,i$ in the form $re^{i\theta}$ where $r > 0$ and $-\pi < \theta < \pi$. **2**

(ii) Find the value of a. **1**

(iii) Find the other 3 roots of the equation in the form $re^{i\theta}$ where $r > 0$ and $-\pi < \theta < \pi$. **3**

(b) **(i)** Use the substitution $t = \tan\dfrac{x}{2}$ to show that $\displaystyle\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{\sin x}\,dx = \ln 3$. **2**

(ii) Use the substitution $u = \pi - x$ to show that $\displaystyle\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{x}{\sin x}\,dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\pi - x}{\sin x}\,dx$. **2**

(iii) Hence find the value of $\displaystyle\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{x}{\sin x}\,dx$. **2**

(c) Let ω be the complex number satisfying $\omega^3 = 1$ and $\text{Im}(\omega) > 0$. **3**
The cubic polynomial

$P(z) = z^3 + az^2 + bz + c$, has zeros $1, -\omega$ and $-\bar{\omega}$.

Find the values for a, b and c in $P(z)$.

End of Question 13

Question 14 (15 marks) Use the Question 14 Writing Booklet.

(a) Consider the equation $z^5 + 1 = 0$.

(i) Draw a sketch of the roots of $z^5 + 1 = 0$ on an Argand Diagram. **1**

(ii) Factor $z^5 + 1$ into irreducible factors with real coefficients. **2**

(iii) Deduce that $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$ and **2**

$$\cos\frac{\pi}{5}\cos\frac{3\pi}{5} = \frac{-1}{4}.$$

(iv) Write a quadratic equation with integer coefficients which has roots **2**

$\cos\frac{\pi}{5}$ and $\cos\frac{3\pi}{5}$.

Hence find the value of $\cos\frac{\pi}{5}$ and $\cos\frac{3\pi}{5}$ as surds.

(b) If $I_n = \int_{-1}^{0} x^n \sqrt{1+x}\, dx \quad n = 0, 1, 2, ...$

(i) Find the value of I_0 **1**

(i) Show that $I_n = \frac{-2n}{2n+3} I_{n-1}$ for $n = 1, 2, 3, ...$ **3**

(ii) Hence find the value of I_2 . **2**

(c) Use proof by contradiction to show that $\log_2 5$ is irrational. **2**

End of Question 14

Question 15 (15 marks) Use the Question 15 Writing Booklet.

(a) Recall that $x+\frac{1}{x}\geq 2$ for any real number $x>0$.
(DO NOT PROVE THIS RESULT)

(i) Prove that $(a+b+c)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c})\geq 9$ for any real numbers $a>0$, $b>0$, $c>0$. **2**

(ii) Prove that $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}$ **3**

for any real numbers $a>0$, $b>0$, $c>0$

by using the result in (i) and transformations of the form $a\rightarrow a+b$.

(b) A particle is moving in a straight line from a fixed point O on the line, so that at time t seconds it has displacement x metres, a velocity v ms^{-1} and an acceleration of $a\,\text{ms}^{-2}$ given by $a=e^{\frac{1}{2}x}$. Initially the particle is at O and moving with a speed of 2 ms^{-1} while slowing down.

(i) Show that $v=-2e^{\frac{1}{4}x}$. **2**

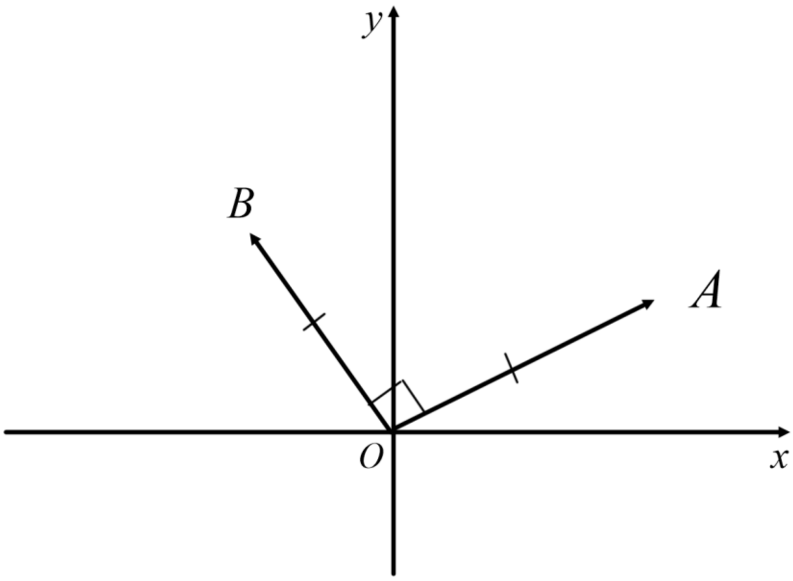
(ii) Find expression for x, v and a in terms of t. **3**

(iii) Describe the subsequent motion of the particle. **1**

Question 15 continues on page 11

Question 15 continues

(c)



In the Argand diagram above, the points A and B represent z_1 and z_2 respectively. $\angle AOB = \frac{\pi}{2}$ and $OA = OB$.

(i) Express z_2 in terms of z_1. **1**

(ii) Copy the diagram and on it draw the locus L_1 of points satisfying $|z - z_1| = |z - z_2|$. **1**

(iii) On your diagram draw the locus L_2 of points satisfying $\arg(z - z_2) = \arg z_1$. **1**

(iv) Find in terms of z_1 the complex number representing the point of intersection of L_1 and L_2. **1**

End of Question 15

Question 16 (15 marks) Use the Question 16 Writing Booklet.

(a) Find $\displaystyle\int_0^{\ln 3} \frac{e^{2x}}{\sqrt{e^x+1}}\,dx$ **4**

(b) A particle of mass *m* kg falls from rest under the influence of gravity g in a medium where the resistance to motion is *mkv* when the particle has velocity v ms^{-1}.

(i) Draw a diagram showing the forces acting on the particle. **1**

(ii) Show that the equation of motion of the particle is $\ddot{x} = k(V - v)$ where V ms^{-1} is the terminal velocity of the particle in this medium, and x metres is the distance fallen in *t* seconds. **2**

(iii) Find in terms of *V* and *k* the time *T* seconds for the particle to attain 50% of its terminal velocity, and the distance fallen in this time. **5**

(iv) What percentage of its terminal velocity will the particle have attained in time 2T seconds? Sketch a graph of *v* against *t* showing this information. **2**

(v) If the particle has reached 87.5% of its terminal velocity in 15 seconds, find the value of *k*. **1**

End of Paper

Caringbah Ext 2 Trial 2020: Multiple Choice Q1-10

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Q1 A, Q2 B, Q3 B, Q4 C, Q5 D, Q6 A, Q7 C, Q8 B, Q9 C, Q10 D

Q1 $z = x + iy$

$\bar{z} = x - iy$

A

$\frac{z - \bar{z}}{2i} = \frac{2iy}{2i} = y$

(2) $\int f(x)\sin x\,dx = -f(x)\cos x + \int f'(x)\cos 3\,dx$

$\therefore f'(x) = 3x^2 \quad f(x) = x^3$

B

(3) $\frac{3x+11}{(x-3)(x+1)} = a(x+1) + b(x-3)$

$a + b = 3$

$a - 3b = 11$

$4b = -8$

$b = -2$

$a = 5$

$\frac{5}{x-3} - \frac{2}{x+1}$ **B**

4	C	No real x satisfies $\|x\| \le -2$. Hence none of A, B, D is a true statement. If $y \le 0$, $\|1\| > y$, and if $y > 0$, $\|y+1\| > y$. Hence C is a true statement.
5	D	$\cos^2\theta + \sin^2\theta + \tan^2\theta = 1 + \tan^2\theta = \sec^2\theta$. Magnitude is $\sec\theta$ $(0 < \theta < \frac{\pi}{2})$
6	A	A is the contra-positive which is logically equivalent (though neither is true)
7	C	$\overrightarrow{AB} = \underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ and $\overrightarrow{AC} = 2\underset{\sim}{i} + (a-1)\underset{\sim}{j} + b\underset{\sim}{k}$ and $\overrightarrow{AC} = \lambda\,\overrightarrow{AB}$ for some real λ. Hence $\frac{2}{1} = \frac{a-1}{-2} = \frac{b}{1}$ $\therefore a = -3$ and $b = 2$
8	B	Period is 2 s and amplitude is 2 m. Hence particle travels 8 m in one oscillation. $100 = 12 \times 8 + 4$ and $12\frac{1}{2}$ oscillations takes 25 s.
9	C	$\ddot{x} = 0$ $\ddot{y} = -g$ $y = -h \Rightarrow t = \sqrt{\frac{2h}{g}}$ $\dot{x} = \sqrt{gh}$ $\dot{y} = -gt$ $x = \sqrt{gh}\,t$ $y = -\frac{1}{2}gt^2$ $\therefore \frac{\dot{y}}{\dot{x}} = \frac{-\sqrt{2gh}}{\sqrt{gh}} = -\sqrt{2}$ $\therefore$ hits ground at angle $\tan^{-1}\sqrt{2}$
10	D	$z^5 - 1 \equiv (z-1)(z^4+z^3+z^2+z+1) \equiv (z-1)(z-\omega)(z-\omega^2)(z-\omega^3)(z-\omega^4)$ $\therefore z^4+z^3+z^2+z+1 \equiv (z-\omega)(z-\omega^2)(z-\omega^3)(z-\omega^4)$ Then putting $z=1$ gives $(1-\omega)(1-\omega^2)(1-\omega^3)(1-\omega^4) = 5$

Q11 a)

i)

Uses compound angle trigonometric identities to establish result	2
Some progress eg. correct procedure with one error	1

Answer

$$\cos 2t+\sin 2t=\sqrt{2}\left(\tfrac{1}{\sqrt{2}}\cos 2t+\tfrac{1}{\sqrt{2}}\sin 2t\right)$$
$$=\sqrt{2}\left(\cos 2t\cos\tfrac{\pi}{4}+\sin 2t\sin\tfrac{\pi}{4}\right)$$
$$=\sqrt{2}\cos\left(2t-\tfrac{\pi}{4}\right)$$

$\therefore x=1+\sqrt{2}\cos\left(2t-\frac{\pi}{4}\right)$

ii)

Finds distance travelled and time taken to first reach O then calculates average speed	3
Substantial progress eg. finds time to reach O and initial position	2
Some progress eg. finds time to reach O	1

Answer

$$x=0\Rightarrow\sqrt{2}\cos\left(2t-\tfrac{\pi}{4}\right)=-1$$
$$\cos\left(2t-\tfrac{\pi}{4}\right)=-\tfrac{1}{\sqrt{2}}$$
$$2t-\tfrac{\pi}{4}=\tfrac{3\pi}{4},\ \tfrac{5\pi}{4},\ldots$$
$$t=\tfrac{\pi}{2},\ \tfrac{3\pi}{4},\ldots$$

$$x=1+\sqrt{2}\cos\left(2t-\tfrac{\pi}{4}\right)$$
$$v=-2\sqrt{2}\sin\left(2t-\tfrac{\pi}{4}\right)$$
$$t=0\Rightarrow x=2,\ v=2$$

In first $\frac{\pi}{2}$ seconds particle moves right from $x=2$ to $x=1+\sqrt{2}$, then left to $x=0$.

Hence average speed during time it takes to first reach O is $\dfrac{\left(1+\sqrt{2}\right)-2+\left(1+\sqrt{2}\right)}{\left(\frac{\pi}{2}\right)}=\dfrac{4\sqrt{2}}{\pi}\approx 1\cdot 80\text{ ms}^{-1}$

b) i)

Writes z_2 in correct form	2
Some progress eg. attempts to realize denominator, but makes one error	1

Answer

$$\frac{12-5i}{3+2i}=\frac{(12-5i)(3-2i)}{3^2+2^2}=\frac{26-39i}{13}\qquad\therefore z_2=2-3i$$

ii)

Shows required result.	1

Answer

$iz_2=i(2-3i)=3+2i=z_1$ Hence rotation of $\overline{OQ}$ anti-clockwise by $\frac{\pi}{2}$ gives $\overline{OP}$. $\therefore\angle POQ=\frac{\pi}{2}$.

iii)

Correct value of z_3	2
Some progress eg. one of real, imaginary parts stated correctly	1

Answer

Interval PQ must be the diameter of the circle. Hence C is the midpoint of PQ. $\therefore z_3=\frac{1}{2}(z_1+z_2)=\frac{5}{2}-\frac{1}{2}i$

c) i)

Uses the given perpendicular lines to deduce the results	2
Some progress eg. correct procedure but poorly explained	1

Answer

$\overline{EB}\perp\overline{AC}$ and O lies on EB. Hence $\overline{OB}.\overline{AC}=0$. $\therefore \underset{\sim}{b}.(\underset{\sim}{c}-\underset{\sim}{a})=0$.

c) i)

Uses the given perpendicular lines to deduce the results	2
Some progress eg. correct procedure but poorly explained	1

Answer

$\overline{EB} \perp \overline{AC}$ and O lies on EB. Hence $\overline{OB}\,.\,\overline{AC} = 0$. $\therefore \underset{\sim}{b}.(\underset{\sim}{c} - \underset{\sim}{a}) = 0$.

Similarly, since $\overline{FC} \perp \overline{AB}$ and O lies on FC, $\underset{\sim}{c}.(\underset{\sim}{b} - \underset{\sim}{a}) = 0$.

ii)

Uses the result from (i) and the properties of dot products to prove the result	2
Some progress eg. use of dot product properties is partially correct	1

Answer

$0 = \underset{\sim}{b}.(\underset{\sim}{c} - \underset{\sim}{a}) = \underset{\sim}{b}.\underset{\sim}{c} - \underset{\sim}{b}.\underset{\sim}{a}$ $\quad \therefore \underset{\sim}{b}.\underset{\sim}{c} = \underset{\sim}{b}.\underset{\sim}{a}$

$0 = \underset{\sim}{c}.(\underset{\sim}{b} - \underset{\sim}{a}) = \underset{\sim}{c}.\underset{\sim}{b} - \underset{\sim}{c}.\underset{\sim}{a}$ $\quad \therefore \underset{\sim}{c}.\underset{\sim}{b} = \underset{\sim}{c}.\underset{\sim}{a}$

Then $\quad \underset{\sim}{a}.\underset{\sim}{b} = \underset{\sim}{a}.\underset{\sim}{c}$

$\therefore \underset{\sim}{a}.(\underset{\sim}{b} - \underset{\sim}{c}) = 0$

Hence $\quad AD \perp CB$

iii)

States geometric property	1

Answer

The altitudes of a triangle are concurrent

Caringbah Ext 2 Trial 2020: Question 12

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a) $\int \frac{x^2}{x^2+1}\,dx = \int\left(1-\frac{1}{x^2+1}\right)dx = x - \tan^{-1}x + c$ /2

b) i)

$$u = x^2$$
$$du = 2x\,dx$$
$$\int \frac{x}{\sqrt{x^2-4}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du$$
$$= u^{\frac{1}{2}} + c$$
$$= \sqrt{x^2-4} + c$$

/2

ii)

$$\int_{\sqrt{5}}^{\sqrt{8}} \frac{x\ln(x^2-4)}{\sqrt{x^2-4}}\,dx$$
$$= \left[\sqrt{x^2-4}\cdot\ln(x^2-4)\right]_{\sqrt{5}}^{\sqrt{8}} - \int_{\sqrt{5}}^{\sqrt{8}} \sqrt{x^2-4}\cdot\frac{2x}{x^2-4}\,dx$$
$$= (2\ln 4 - 1\ln 1) - \int_{\sqrt{5}}^{\sqrt{8}} \frac{2x}{\sqrt{x^2-4}}\,dx$$
$$= 2\ln 4 - 2\left[\sqrt{x^2-4}\right]_{\sqrt{5}}^{\sqrt{8}}$$
$$= 4\ln 2 - 2$$

/3

c) i)

(c) (i) Find the parametric equations of the line l passing through $A=(2,-1,3)$ parallel to $\underset{\sim}{v} = \begin{pmatrix}4\\3\\-2\end{pmatrix}$ 2

$$\underset{\sim}{l} = \underset{\sim}{a} + \lambda \underset{\sim}{v}$$
$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\-1\\3\end{pmatrix} + \lambda\begin{pmatrix}4\\3\\-2\end{pmatrix}$$

/2

$$x = 2 + 4\lambda$$
$$y = -1 + 3\lambda$$
$$z = 3 - 2\lambda$$

(ii) Show that the point $B=(10,5,-1)$ lies on this line. 2

$10 = 2 + 4\lambda$ – ①

$5 = -1 + 3\lambda$ – ②

$-1 = 3 - 2\lambda$ – ③

1

① + ② gives $5 = 3 + \lambda \therefore \lambda = 2$

Sub in ③ $-1 = 3 - 2\times 2 = -1$

1

$\therefore (10, 5, -1)$ lies on this line

(d)	If $\underset{\sim}{a} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{b} = -\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$ find a unit vector perpendicular to both $\underset{\sim}{a}$ and $\underset{\sim}{b}$.	**2**

Unit vector $\underset{\sim}{v} = a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$

$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} = 0$ $\quad \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = 0$

$3a + 2b + 2c = 0$ - ①

$-a + b + c = 0$ - ②

② × 2

$-2a + 2b + 2c = 0$ - ③

∴ ① - ③

$5a = 0$

$a = 0$

∴ $b = -c$

unit vector means

$a^2 + b^2 + c^2 = 1$

∴ $2b^2 = 1$

$b = \pm \frac{1}{\sqrt{2}} \quad c = \mp \frac{1}{\sqrt{2}}$

∴ $\underset{\sim}{v} = \pm\left(\frac{1}{\sqrt{2}} j - \frac{1}{\sqrt{2}} k\right)$

d)

Uses log laws to prove statement is true	2
Some progress eg. applies log laws but one error or incomplete explanation	1

Answer

Using log laws, for $\forall a > 0,\ \forall b > 0$, $\log_{\frac{1}{a}} \frac{1}{b} = \dfrac{\log_a \frac{1}{b}}{\log_a \frac{1}{a}} = \dfrac{-\log_a b}{-1} = \log_a b$

Caringbah Ext 2 Trial 2020: Question 13

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a) i)

Correct expression	2
Some progress eg. finds the modulus	1

Answer

$$1+\sqrt{3}\,i=2\left(\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right)=2\left(\cos\tfrac{\pi}{3}+i\sin\tfrac{\pi}{3}\right)=2e^{\frac{\pi}{3}i}$$

ii)

Correct value of a	1

Answer

$$\left(2e^{\frac{\pi}{12}i}\right)^4=2ae^{\frac{\pi}{3}i}$$
$$16e^{\frac{\pi}{3}i}=2ae^{\frac{\pi}{3}i}$$
$$\therefore a=8$$

iii)

States other 3 roots in required form	3
Substantial progress eg. correct except that θ is not in specified domain	2
Some progress eg. finds one other root	1

Answer

$$\arg z^4=\tfrac{\pi}{3}+2k\pi,\quad k=0,\ \pm1,\ \pm2,\ldots$$
$$\arg z=\tfrac{\pi}{12}+k\tfrac{\pi}{2}$$
$$|z|=2$$

Other roots are
$$2e^{\left(\frac{\pi}{12}+\frac{\pi}{2}\right)i}=2e^{\frac{7\pi}{12}i}$$
$$2e^{\left(\frac{\pi}{12}-\frac{\pi}{2}\right)i}=2e^{-\frac{5\pi}{12}i}$$
$$2e^{\left(\frac{\pi}{12}-\pi\right)i}=2e^{-\frac{11\pi}{12}i}$$

b) i)

$$t=\tan\tfrac{x}{2}$$
$$dt=\tfrac{1}{2}\sec^2\tfrac{x}{2}\,dx$$
$$2\,dt=\left(1+\tan^2\tfrac{x}{2}\right)dx$$
$$\frac{2}{1+t^2}\,dt=dx$$
$$x=\tfrac{\pi}{3}\Rightarrow t=\tfrac{1}{\sqrt{3}}$$
$$x=\tfrac{2\pi}{3}\Rightarrow t=\sqrt{3}$$

$$\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\frac{1}{\sin x}\,dx=\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}\frac{1+t^2}{2t}\cdot\frac{2}{1+t^2}\,dt$$
$$=\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}\frac{1}{t}\,dt$$
$$=\left[\ln t\right]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}$$
$$=\tfrac{1}{2}\ln 3-\left(-\tfrac{1}{2}\ln 3\right)$$
$$\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\frac{1}{\sin x}\,dx=\ln 3$$

/2

ii)

$$u=\pi-x\ \Rightarrow du=-dx$$
$$\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\frac{x}{\sin x}\,dx=\int_{\frac{2\pi}{3}}^{\frac{\pi}{3}}\frac{\pi-u}{\sin(\pi-u)}\cdot -du$$
$$=\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\frac{\pi-u}{\sin u}\,du$$
$$\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\frac{x}{\sin x}\,dx=\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\frac{\pi-x}{\sin x}\,dx$$

/2

ii)

Let $I=\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{x}{\sin x}\,dx=\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\pi-x}{\sin x}\,dx$

Then $$2I=\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\left(\frac{x}{\sin x}+\frac{\pi-x}{\sin x}\right)\,dx$$
$$=\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\frac{\pi}{\sin x}\,dx$$
$$=\pi\ln 3$$

Hence $I=\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\frac{x}{\sin x}\,dx=\frac{\pi}{2}\ln 3$

c) $P(z)=z^3+az^2+bz+c$

METHOD 1

Since ω is the complex cube root of unity and $\text{Im}(\omega)>0$,

$$\omega=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$$
$$\bar{\omega}=\cos\frac{2\pi}{3}-i\sin\frac{2\pi}{3}$$

Hence $\omega+\bar{\omega}=2\cos\frac{2\pi}{3}$
$=-1$

and $\omega\bar{\omega}=|\omega|^2$
$=1$

$$\therefore p(z)=(z-1)(z+\omega)(z+\bar{\omega})$$
$$=(z-1)\left[z^2+(\omega+\bar{\omega})z+\omega\bar{\omega}\right]$$
$$=(z-1)(z^2-z+1)$$
$$=z^3-2z^2+2z-1.$$

METHOD 2

From Method 1, $\omega+\bar{\omega}=-1$ and $\omega\bar{\omega}=1$

Sum of roots

$$-a=1+(-\omega)+(-\bar{\omega})$$
$$=1-(\omega+\bar{\omega})$$
$$=1-(-1)$$
$$=2$$

$\therefore a=-2$

Sum of roots taken two at a time

$$b=1\times(-\omega)+1\times(-\bar{\omega})+(-\omega)(-\bar{\omega})$$
$$=-(\omega+\bar{\omega})+\omega\bar{\omega}$$
$$=-(-1)+1$$
$$=2$$

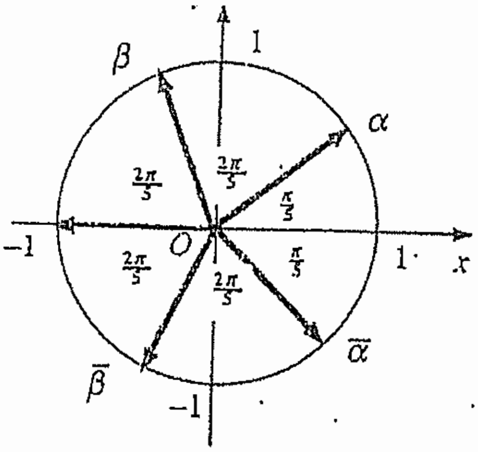
Product of roots

$$-c=1\times(-\omega)\times(-\bar{\omega})$$
$$=\omega\bar{\omega}$$
$$=1$$

$\therefore c=-1$

Hence $p(z)=z^3-2z^2+2z-1.$

a) i)



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ii)

$$\alpha = \cos\tfrac{\pi}{5} + i\sin\tfrac{\pi}{5} \Rightarrow \alpha\overline{\alpha} = |\alpha|^2 = 1\ ,\ \alpha + \overline{\alpha} = 2\cos\tfrac{\pi}{5}$$

$$\beta = \cos\tfrac{3\pi}{5} + i\sin\tfrac{3\pi}{5} \Rightarrow \beta\overline{\beta} = |\beta|^2 = 1,\ \ \beta + \overline{\beta} = 2\cos\tfrac{3\pi}{5}$$

$$(z-\alpha)(z-\overline{\alpha}) = z^2 - (\alpha+\overline{\alpha})z + \alpha\overline{\alpha}$$

$$z^5 + 1 = (z+1)(z-\alpha)(z-\overline{\alpha})(z-\beta)(z-\overline{\beta})$$

$$\therefore z^5 + 1 = (z+1)\left(z^2 - 2\cos\tfrac{\pi}{5}\cdot z + 1\right)\left(z^2 - 2\cos\tfrac{3\pi}{5}\cdot z + 1\right)$$

1/2

iii)

Equating coefficients of z:

$0 = 1 - 2\cos\frac{\pi}{5} - 2\cos\frac{3\pi}{5}$ (1)

Equating coefficients of z^2:

$0 = 1 + 1 + 4\cos\frac{\pi}{5}\cos\frac{3\pi}{5} - 2\cos\frac{\pi}{5} - 2\cos\frac{3\pi}{5}$ (2)

(1) $\Rightarrow$ $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$

(2) − (1) $\Rightarrow$ $\cos\frac{\pi}{5}\cdot\cos\frac{3\pi}{5} = -\frac{1}{4}$

1/2

iv)

$\cos\frac{\pi}{5}$ and $\cos\frac{3\pi}{5}$ are roots of

$4x^2 - 2x - 1 = 0$

$$x = \frac{2 \pm \sqrt{20}}{8} = \frac{1 \pm \sqrt{5}}{4}$$

$$\cos\tfrac{\pi}{5} > 0 \Rightarrow \cos\tfrac{\pi}{5} = \frac{1+\sqrt{5}}{4}$$

$$\cos\tfrac{3\pi}{5} < 0 \Rightarrow \cos\tfrac{3\pi}{5} = \frac{1-\sqrt{5}}{4}$$

1/2

b)

Correct answer	1

i)

Answer

$$I_1 = \int_0^1 \frac{1}{1+x^2}\,dx = \left[\tan^{-1}x\right]_0^1 = \frac{\pi}{4}$$

i)

$$I_1=\int_0^1 \frac{1}{1+x^2}\,dx=\left[\tan^{-1}x\right]_0^1=\frac{\pi}{4}$$

ii)

Applies integration by parts and rearranges result into required form	3
Substantial progress eg. correct procedure but with one error in execution	2
Some progress eg. applies integration by parts	1

Answer

$$I_n=\int_0^1 1\cdot\frac{1}{(1+x^2)^n}\,dx$$

$$=\left[\frac{x}{(1+x^2)^n}\right]_0^1-\int_0^1 x\cdot\frac{-2nx}{(1+x^2)^{n+1}}\,dx$$

$$=2^{-n}+2n\int_0^1\frac{(1+x^2)-1}{(1+x^2)^{n+1}}\,dx$$

$$I_n=2^{-n}+2n\left\{I_n-I_{n+1}\right\}$$

$$2nI_{n+1}=2^{-n}+(2n-1)I_n$$

iii)

Applies recurrence formula to evaluate as required	2
Some progress eg. one error in application of recurrence formula	1

Answer

$$4I_3=2^{-2}+3I_2=2^{-2}+\tfrac{3}{2}\left(2^{-1}+I_1\right)=1+\tfrac{3\pi}{8}\qquad\therefore I_3=\frac{8+3\pi}{32}$$

c)

Uses the definition of a rational number to construct a proof by contradiction	2
Some progress eg. quotes the condition for $\log_2 5$ to be rational	1

Answer

$5>1\ \therefore \log_2 5>0$. Hence $\log_2 5$ is rational $\Rightarrow \exists$ positive integers p, q with no common factor such that $\log_2 5=\dfrac{p}{q}$. Then

$$q\log_2 5=p$$

$$\therefore\ \log_2 5^q=p$$

$$\therefore\ 5^q=2^p$$

But 5 and 2 are prime numbers, so this last statement cannot be true for any positive integers p, q. Hence by contradiction $\log_2 5$ is irrational.

Caringbah Ext 2 Trial 2020: Question 15

Monday, 27 July 2020 8:34 PM

a) i)

Expands and regroups to establish result	2
Some progress eg. expands	1

Answer

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=3+\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{c}{a}+\frac{a}{c}\right)$$
$$\geq 3+2+2+2$$
$$=9$$

ii)

Makes an appropriate transformation to establish required result	3
Substantial progress eg. appropriate transformation, and almost completes proof	2
Some progress eg. appropriate transformation with some attempt to expand	1

Answer

$a \to a+b$

$b \to b+c$ gives

$c \to c+a$

$$2(a+b+c)\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)\geq 9$$
$$\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)+\left(\frac{a}{a+b}+\frac{b}{a+b}\right)+\left(\frac{b}{b+c}+\frac{c}{b+c}\right)+\left(\frac{c}{c+a}+\frac{a}{c+a}\right)\geq\frac{9}{2}$$
$$\therefore \left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)+1+1+1\geq\frac{9}{2}$$
$$\therefore \quad \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}$$

b) i)

Uses appropriate expression for a and integration to produce required expression for v	2
Some progress eg. correct procedure but neglects to explain –ve sign	1

Answer

Initially $v<0$ since $a>0$ and particle is slowing down.

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx}=e^{\frac{1}{2}x}$$
$$\frac{1}{2}v^2=2e^{\frac{1}{2}x}+c$$

$$\left.\begin{array}{l} t=0 \\ x=0 \\ v=-2 \end{array}\right\} \Rightarrow \begin{array}{l} c=0 \\ v^2=4e^{\frac{1}{2}x} \\ v=-2e^{\frac{1}{4}x} \end{array}$$

ii)

Finds x in terms of t by integration, then states v and a in terms of t	3
Substantial progress eg. finds x in terms of t by integration	2
Some progress eg. uses integration to find x in terms of t, but makes one error	1

Answer

$$\frac{dx}{dt}=-2e^{\frac{1}{4}x}$$

ii)

Finds x in terms of t by integration, then states v and a in terms of t	3
Substantial progress eg. finds x in terms of t by integration	2
Some progress eg. uses integration to find x in terms of t, but makes one error	1

Answer

$$\frac{dx}{dt} = -2e^{\frac{1}{4}x}$$

$$\int e^{-\frac{1}{4}x}\,dx = \int -2\,dt$$

$$-4e^{-\frac{1}{4}x} = -2t + c$$

$$\left.\begin{matrix} t=0 \\ x=0 \end{matrix}\right\} \Rightarrow \quad c=-4, \quad e^{-\frac{1}{4}x} = \frac{t+2}{2}$$

$$x = 4\ln\left(\frac{2}{2+t}\right)$$

$$v = -2e^{\frac{1}{4}x}$$

$$\therefore v = \frac{-4}{2+t}$$

$$a = e^{\frac{1}{2}x} = \left(e^{\frac{1}{4}x}\right)^2$$

$$\therefore a = \frac{4}{(2+t)^2}$$

iii)

Correct description	1

Answer

Particle continues to move in the initial direction of travel without bound, but slowing down at an ever decreasing rate with velocity and acceleration both approaching 0.

c) i)

Correct expression	1

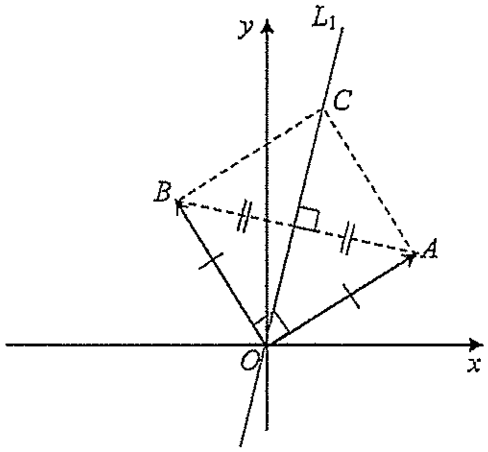
Answer $z_2 = i\,z_1$

ii)

Sketches the perpendicular bisector of AB	1

Answer

L_1 is the perpendicular bisector of AB. If C is such that $OACB$ is a square, then L_1 is the line OC.

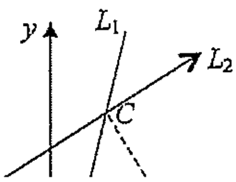


iii)

Sketches a ray from B parallel to vector $\overline{OA}$	1

Answer

L_2 is a ray from B (excluding the point B) that is parallel to the vector $\overline{OA}$. This ray will pass through the vertex C of the square $OACB$.

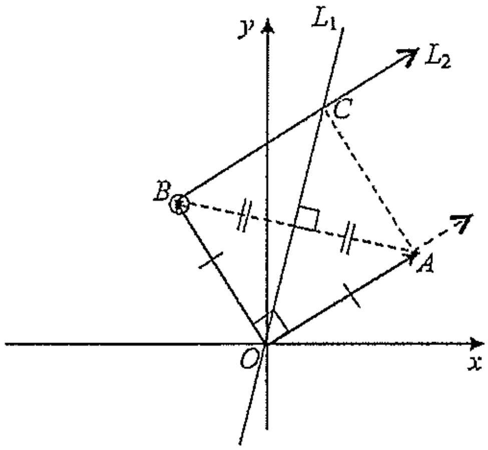


iii)

Sketches a ray from B parallel to vector $\overline{OA}$	1

Answer

L_2 is a ray from B (excluding the point B) that is parallel to the vector $\overrightarrow{OA}$. This ray will pass through the vertex C of the square $OACB$.



iv)

Correct answer	1

Answer

The intersection of L_1 and L_2 is the vertex C of the square OACB and C represents $z_1 + z_2 = (1+i)z_1$.

a) $\displaystyle\int_0^{\ln 3} \frac{e^{2x}}{\sqrt{e^x+1}}\,dx$

Let $u^2 = e^x + 1$, $2u\,du = e^x\,dx$ (1)

$x = 0$, $u = \sqrt{2}$

$x = \ln 3$, $u = \sqrt{4} = 2$ (1)

$$= \int_{\sqrt{2}}^{2} \frac{(u^2-1)\,2u\,du}{u}$$

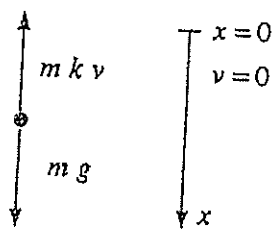
$$= \int_{2}^{\sqrt{2}} 2u^2 - 2\,du \quad 1$$

$$= \left[\frac{2u^3}{3} - 2u\right]_{\sqrt{2}}^{2}$$

$$= \frac{16}{3} - 4 - \left(\frac{4\sqrt{2}}{3} - 2\sqrt{2}\right)$$

$$= \frac{4+2\sqrt{2}}{3} \quad 1$$

b) i) *Forces on particle*



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ii) $m\ddot{x} = mg - mkv$

$\ddot{x} = k\left(\frac{g}{k} - v\right)$

As $v \to \frac{g}{k}$, $\ddot{x} \to 0$.

Hence terminal velocity $V = \frac{g}{k}$

$\therefore \ddot{x} = k(V - v)$

/2

iii) $v\frac{dv}{dx} = k(V - v)$

$\frac{dv}{dx} = k\frac{V-v}{v}$

$\frac{dv}{dt} = k(V - v)$

$-k\frac{dt}{dv} = \frac{-1}{V-v}$

/1

$$\frac{dv}{dx} = k\frac{V-v}{v}$$

$$-k\frac{dx}{dv} = \frac{-v}{V-v}$$

$$= 1 + V\frac{-1}{V-v}$$

$$-kx = v + V\ln\{(V-v)B\},\ B \text{ constant}$$

$$\left.\begin{matrix} x=0 \\ v=0 \end{matrix}\right\} \Rightarrow B = \frac{1}{V}$$

$$-kx = v + V\ln\left(\frac{V-v}{V}\right)$$

$$x = \frac{1}{k}\left\{-v + V\ln\left(\frac{V}{V-v}\right)\right\}$$

/2

$$-k\frac{dt}{dv} = \frac{-1}{V-v}$$

$$-kt = \ln\{(V-v)A\},\ A \text{ constant}$$

$$\left.\begin{matrix} t=0 \\ v=0 \end{matrix}\right\} \Rightarrow A = \frac{1}{V}$$

$$-kt = \ln\left(\frac{V-v}{V}\right)$$

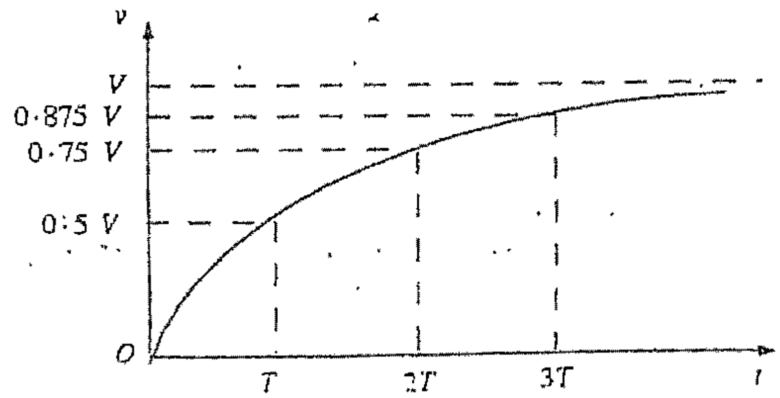
$$t = \frac{1}{k}\ln\left(\frac{V}{V-v}\right)$$

/2

Particle attains 50% of terminal velocity when $v = \frac{1}{2}V$, $t = T = \frac{\ln 2}{k}$ and $x = \frac{V}{2k}(2\ln 2 - 1)$.

/1

iv) $(V-v) = Ve^{-kt} \Rightarrow (V-v)$ is decaying exponentially. Hence $(V-v)$ halves every T seconds.
Hence particle reaches 75% of terminal velocity in $2T$ seconds.



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v) $v = 87\cdot5\%$ of V after $3T$ seconds.

$$3T = 15 \Rightarrow T = \frac{\ln 2}{k} = 5$$

$$\therefore k = \tfrac{1}{5}\ln 2$$

/1